CIRCULAR HOLE UNDER DISCONTINUOUS TANGENTIAL STRESSES*

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ABSTRACT. In this problem, the stresses and displacements due to discontinuous tangential stresses applied in a specified manner at the boundary of a circular hole in an infinite plate have been studied. It may be noted that the boundary tractions have a resultant. The problem has been solved with the help of integral equation method coupled with Fourier series representation of the boundary conditions.

Consider a circular hole in an infinite plate and let the tangential stresses be applied at the boundary of the hole in the manner as shown in the adjoining figure 1. Let the radius of the circular hole be R with its centre at the origin so that the equation of the boundary of the hole may be written as $\sigma\bar{\sigma} = 1$, where σ is the complex coordinate of any boundary point.

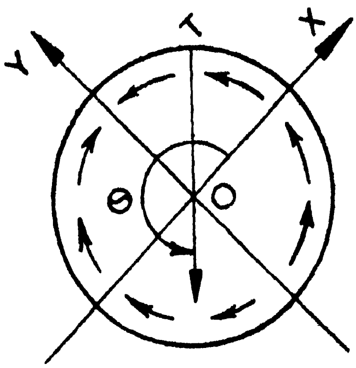

Figure 1.

As is well-known the solution of any two dimensional problem in elasticity can be obtained if two analytic functions $\phi(z)$ and $\psi(z)$ of the variable $z = x+iy$ are known. The boundary conditions on the circle may be written as (Muskhelishvili, 1963 and Sokolnikoff, 1956)

$$\phi(\sigma)+\sigma\overline{\phi'(\sigma)}+\overline{\psi(\sigma)} = f_1+if_2 = f$$

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or taking its conjugate, the boundary conditions are

$$\overline{\phi(\sigma)}+\bar{\sigma}\phi'(\sigma)+\psi(\sigma)=f_1-if_2=\bar{f}$$

where

$$f_1+if_2=f=i\int^{s}(P_{nx}+iP_{ny})ds \qquad .. \quad (1)$$

and P_{nx}, P_{ny} are the components of boundary tractions in positive x and y directions on the surface whose outward drawn normal is n; S is the arcual distance of a point on the boundary, measured from a fixed point.

One has to be careful at this stage, θ is the angle which the outward drawn normal (in the present case, this normal points towards the centre) makes with the x-axis in the anti-clockwise direction. It is found that, if T be boundary tangential traction,

$$P_{nx}=T\,\text{Sin}\,\theta, \qquad P_{ny}=-T\,\text{Cos}\,\theta, \qquad 0<\theta<\frac{\pi}{2};$$

$$P_{nx}=-T\,\text{Sin}\,\theta, \qquad P_{ny}=T\,\text{Cos}\,\theta, \qquad \frac{\pi}{2}<\theta<\pi;$$

$$P_{nx}=-T\,\text{Sin}\,\theta, \qquad P_{ny}=T\,\text{Cos}\,\theta, \qquad \pi<\theta<\frac{3\pi}{2};$$

$$P_{nx}=T\,\text{Sin}\,\theta, \qquad P_{ny}=-T\,\text{Cos}\,\theta. \qquad \frac{3\pi}{2}<\theta<2\pi.$$

Substituting these values in (1) it may be seen that

$$\begin{aligned} f_1+if_2 &= iRT[1-e^{i\theta}], && \text{in the first quadrant,} \\ &= iRT[1-2i+e^{i\theta}], && \text{in the second quadrant,} \\ &= iRT[1-2i+e^{i\theta}], && \text{in the third quadrant,} \\ &= iRT[1-4i-e^{i\theta}], && \text{in the fourth quadrant.} \end{aligned}$$

The function f_1+if_2 is expanded in terms of a Fourier series.

If

$$f_1+if_2=\sum_{-\infty}^{\infty}a_n e^{in\theta}$$

we easily obtain that

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}(f_1+if_2)e^{-in\theta}d\theta, \qquad \text{for } n\neq 1$$

$$=\frac{iRT}{n\pi}\left[\frac{1}{n-1}(e^{-in\pi/2}+e^{-i.3\pi/2})+2\right].$$

The values of some of the non-zero coefficients are :

$$a_0 = \frac{iRT}{\pi}\,[\pi-2-2i\pi], \qquad a_1 = \frac{2iRT}{\pi}, \qquad a_2 = 0,$$

$$a_3 = \frac{2iRT}{3\pi}, \qquad a_4 = \frac{2iRT}{3\pi}, \qquad a_5 = \frac{2iRT}{5\pi}, \qquad a_6 = \frac{4iRT}{3\cdot 5\pi},$$

$$a_7 = \frac{2iRT}{7\pi}, \qquad a_8 = \frac{2iRT}{7\pi}, \qquad a_9 = \frac{2iRT}{9\pi}, \qquad a_{10} = \frac{8iRT}{5\cdot 9\pi},$$

$$a_{11} = \frac{2iRT}{11\pi}, \qquad a_{12} = \frac{2iRT}{11\pi}, \qquad a_{13} = \frac{2iRT}{13\pi}, \qquad a_{14} = \frac{12iRT}{7\cdot 13\pi}, \text{ etc.}$$

Also,

$$a_{-1} = -\frac{2iRT}{\pi}, \quad a_{-2} = -\frac{4iRT}{3\pi}, \quad a_{-3} = -\frac{2iRT}{3\pi}, \quad a_{-4} = -\frac{2iRT}{5\pi},$$

$$a_{-5} = -\frac{2iRT}{5\pi}, \quad a_{-6} = -\frac{8iRT}{3\cdot 7\pi}, \quad a_{-7} = -\frac{2iRT}{7\pi}, \quad a_{-8} = -\frac{2iRT}{9\pi},$$

$$a_{-9} = -\frac{2iRT}{9\pi}, \quad a_{-10} = -\frac{12iRT}{5\cdot 11\pi}, \quad a_{-11} = -\frac{2iRT}{11\pi}, \quad a_{-12} = -\frac{2iRT}{13\pi},$$

$$a_{-13} = -\frac{2iRT}{13\pi}, \quad a_{-14} = -\frac{16iRT}{7\cdot 15\pi}, \quad a_{-15} = -\frac{2iRT}{15\pi}, \quad a_{-16} = -\frac{2iRT}{17\pi},$$

etc.

In the case of simply-connected infinite region $\phi(z)$ and $\psi(z)$ are given by Muskhelishvili, (1963) and Sokolinkoff, (1956).

$$\phi(z) = \Gamma Rz - \frac{X+iY}{2\pi(1+K)} \log z + \phi_0(z),$$

$$\psi(z) = \Gamma' Rz + \frac{K(X-iY)}{2\pi(1+K)} \log z + \psi_0(z), \qquad \ldots \quad (2)$$

where Z is any point in the region; Γ and Γ' are the external stresses at infinity; $X+iY$ is the resultant force on the interior boundary; and $\phi_0(z)$, $\psi_0(z)$ are two analytic functions in the infinite region.

In the present problem the stresses at infinite are zero, hence

$$\Gamma = \Gamma' = 0$$

Also $X = 0$ and $Y = 4\ RT$. At this stage it may be noted that $X = \oint P_{nx} ds$ as $Y = \oint P_{ny} ds$. As the outward normal points to the centre, s is to be measured in the clockwise direction.

Substituting the values of Γ, Γ', X and Y in (2) the above equations yield

$$\phi(z) = -\frac{2iRT}{\pi(1+K)} \log z + \phi_0(z),$$

$$\psi(z) = -\frac{2iKRT}{\pi(1+K)} \log z + \psi_0(z).$$

At this stage, it is convenient to write $z = R\rho$, thus mapping the circle of radius R in the z-plane to a circle of unit radius in the ρ-plane. It is trivial to speak that the outside region in the z-plane is being mapped to the outside region in the ρ-plane.

The functions $\phi_0(\rho)$ and $\psi_0(\rho)$ for corresponding to $\phi_0(z)$ and $\psi_0(z)$ may be obtained as follows (Mushelishvili, 1963).

$$\phi_0(\rho) = -\frac{1}{2\pi i} \int_\gamma \frac{f_0 d\sigma}{\sigma - \rho}, \qquad \text{.. (3)}$$

$$\psi_0(\rho) = -\frac{1}{2\pi i} \int_\gamma \frac{\bar{f}_0 d\sigma}{\sigma - \rho} - \frac{1}{\rho} \phi'_0(\rho), \qquad \text{.. (4)}$$

where

$$f_0 = f + \frac{2iRT}{\pi} \log \sigma - \frac{2iRT}{\pi(1+K)} \sigma^2.$$

Also f is given by

$$f = \sum_0^\infty a_n e^{in\theta} + \sum_1^\infty a_{-n} e^{-in\theta}$$

Substituting the value of f_0 in (3), it is seen that

$$\phi_0(\rho) = -\frac{2iRT}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \frac{1}{2n(2n+1)} \cdot \frac{1}{\rho^{2n}}.$$

Thus

$$\phi(\rho) = -\frac{2iRT}{\pi(1+K)} \log \rho - \frac{2iRT}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \frac{1}{2n(2n+1)} \cdot \frac{1}{\rho^{2n}}. \qquad \text{.. (5)}$$

To calculate $\psi(\rho)$, the value of $\bar{f}_0$ is calculated by

$$\bar{f}_0 = \bar{f} + \frac{2iRT}{\pi} \log \sigma + \frac{2iRT}{\pi(1+K)} \frac{1}{\sigma^2}$$

where

$$\bar{f} = \sum_0^\infty a_n e^{in\theta} + \sum_1^\infty a_{-n} e^{-ni\theta}.$$

Substituting the value of $\bar{f}_0$ in (4), it is found that

$$\psi_0(\rho) = \frac{2iRT}{\pi} \left[\frac{3+K}{2(1+K)} \cdot \frac{1}{\rho^2} + \sum_{n=2}^\infty (-1)^{n+1} \frac{2n+1}{(2n-1)2n} \cdot \frac{1}{\rho^{2n}} \right].$$

Thus

$$\psi(\rho) = -\frac{2iKRT}{\pi(1+K)} \log \rho + \frac{2iRT}{\pi} \left[\frac{3+K}{2(1+K)} \cdot \frac{1}{\rho^2} + \sum_{n=2}^\infty (-1)^{n+1} \frac{2n+1}{(2n-1)2n} \cdot \frac{1}{\rho^{2n}} \right]. \quad \text{.. (6)}$$

Now taking

$$\Phi(z) = \Phi(\rho) = \frac{\varphi'(\rho)}{\omega'(\rho)} = \frac{\varphi'(\rho)}{R}$$

$$\Phi(\rho) = -\frac{2iT}{\pi} \left[\frac{1}{1+K} \frac{1}{\rho} + \sum_{n=1}^\infty (-1)^n \frac{1}{2n+1} \cdot \frac{1}{\rho^{2n+1}} \right],$$

or

$$\Phi(z) = -\frac{2iT}{\pi} \left[\frac{1}{1+K} \frac{R}{Z} + \sum_{n=1}^\infty (-1)^n \frac{1}{2n+1} \cdot \left(\frac{R}{Z}\right)^{2n+1} \right]. \quad \text{.. (7)}$$

Again $$\psi_1(Z) = \Psi(\rho) = \frac{\psi'(\rho)}{R}$$

Or

$$\psi(\rho) = \frac{2iT}{\pi} \left[-\frac{K}{1+K} \frac{1}{\rho} + \left\{ -\frac{3+K}{1+K} \frac{1}{\rho^3} + \sum_{n=2}^\infty (-1)^n \frac{2n+1}{2n-1} \cdot \frac{1}{\rho^{2n+1}} \right\} \right]$$

or

$$\psi(Z) = \frac{2iT}{\pi} \left[-\frac{K}{1+K} \frac{R}{Z} - \frac{3+K}{1+K} \left(\frac{R}{Z}\right)^3 + \sum_{n=2}^\infty (-1)^n \frac{2n+1}{2n-1} \left(\frac{R}{Z}\right)^{2n+1} \right] \quad \text{.. (8)}$$

Now taking $z = re^{i\theta}$ and $\hat{z} = re^{-i\theta}$, we obtain

$$\widehat{rr}+\widehat{\theta\theta} = 4Re\Phi(z)$$

$$= \frac{8T}{\pi}\left[-\frac{1}{1+K}\frac{R}{r}\operatorname{Sin}\theta+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2n+1}\left(\frac{R}{r}\right)^{2n+1}\operatorname{Sin}(2n+1)\theta\right], \quad \ldots \quad (9)$$

And

$$\widehat{\theta\theta}-\widehat{rr}+2i\widehat{r\theta} = 2[\widehat{Z}\Phi'(z)+\psi(z)]e^{2i\theta}$$

$$= \frac{4iT}{\pi}\left[\left(\frac{1}{1+K}\frac{R}{r}-\frac{3+K}{1+K}\frac{R^3}{r^3}\right)e^{-i\theta}-\frac{K}{1+K}\frac{R}{r}e^{i\theta}\right.$$

$$\left.+\sum_{n=1}^{\infty}(-1)^n\left\{\left(\frac{R}{r}\right)^{2n+1}-\frac{2n+3}{2n+1}\left(\frac{R}{r}\right)^{2n+3}\right\}e^{-i(2n+1)\theta}\right]. \quad \ldots \quad (10)$$

Separating real and imaginary parts from (10) it is found that

$$\widehat{rr} = -\frac{2T}{\pi}\left[-\frac{2K}{1+K}\left(\frac{R}{r}-\frac{R^3}{r^3}\right)\sin\theta+\frac{R}{r}\left(1-\frac{R^2}{r^2}\right)^2\left\{\frac{\sin\theta}{1+\frac{2R^2}{r^2}\operatorname{Cos}2\theta+\frac{R^4}{r^4}}\right.\right.$$

$$\left.\left.+\tfrac{1}{4}\log\frac{2Rr}{r^2+R^2}+\tfrac{1}{2}\log\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}\right], \quad \ldots \quad (11)$$

$$\widehat{\theta\theta} = \frac{2T}{\pi}\left[\frac{2K}{1+K}\left(\frac{R}{r}+\frac{R^3}{r^3}\right)\sin\theta+\frac{R}{r}\left(1-\frac{R^2}{r^2}\right)^2\frac{\sin\theta}{1+\frac{2R^2}{r^2}\cos 2\theta+\frac{R^4}{r^4}}\right.$$

$$\left.-\left(1+\frac{R^2}{r^2}\right)\left\{\tfrac{1}{2}\log\frac{2Rr}{r^2+R^2}+\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right\}\right], \quad \ldots \quad (12)$$

$$\widehat{r\theta} = \frac{2T}{\pi}\left[-\frac{2K}{1+K}\left(\frac{R}{r}-\frac{R^3}{r^3}\right)\cos\theta+\frac{R}{r}\left(1-\frac{R^4}{r^4}\right)\frac{\cos\theta}{1+\frac{2R^2}{r^2}\cos 2\theta+\frac{R^4}{r^4}}\right.$$

$$\left.-\frac{R^2}{r^2}\tan^{-1}\frac{2Rr}{r^2-R^2}\cos\theta\right]. \quad \ldots \quad (13)$$

On the boundary of the hole where $R = r$, the results are :

$$\widehat{rr} = 0,$$

$$\widehat{\theta\theta} = \frac{8T}{\pi}\left[\frac{K}{1+K}\sin\theta - \tfrac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$$

$$\widehat{r\theta} = -T. \tag{14}$$

Because of discontinuities at $\theta = \pm\pi/2$, the normal stress $\widehat{\theta\theta}$ exhibits infinity at these points.

For the displacement components we note that

$$2\mu(v_r + iv_\theta) = e^{-i\theta}[K\varphi(z) - z\overline{\varphi'(z)} + \overline{\psi(z)}] \quad .. \quad (15)$$

The displacement components are given by substituting the values of $\phi(z)$ and $\psi(z)$ in the above expression.

REFERENCES

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Sokolinkoff, I. S. 1956, *Mathematical Theory of Elasticity*, McGraw Hill, London.